

Spin in Quantum Field Theory

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Abstract

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1 From Quantum Mechanics to Field Theory

Even though everybody learns about spin in their childhood in the context of nonrelativistic quantum mechanics, many of the more interesting dynamical features of spin are only introduced in relativistic quantum field theory. In these lectures, which were originally addressed to an audience of (mostly) condensed-matter physicists, I discuss some relevant aspects of spin dynamics in quantum field theory by showing their origin in quantum mechanics. In the first lecture, after a brief discussion of the way spin appears in nonrelativistic (Galilei invariant) or relativistic (Lorentz invariant) dynamics, I show how the spin–statistics connection can be obtained with minimal assumptions in nonrelativistic quantum mechanics, without invoking relativity or field theory. In the second lecture I show how spin can be quantized in a path–integral approach with no need for introducing quantum fields. In the third lecture I discuss the dynamics of relativistic spinning particles and show that its quantization can be described without using anticommuting variables. A fourth lecture was devoted to the quantum breaking of chiral symmetry – the axial anomaly – and its origin in the structure of the spectrum of the Dirac operator, but since this subject is already covered in many classic lectures [1] it will not be covered here. We will see that even though the standard methods of quantum field theory are much more practical for actual calculations, a purely quantum–mechanical approach helps in understanding the meaning of field–theoretic concepts.

2 Spin and Statistics

2.1 The Galilei Group and the Lorentz Group

In both relativistic and non-relativistic dynamics we can understand the meaning of quantum numbers in terms of the symmetries of the Hamiltonian and the Lagrangian and associated action. Indeed, the invariance of the Hamiltonian determines the spectrum of physical states: eigenstates of the Hamiltonian are classified by the eigenvalues of operators which commute with it, and this gives the set of observables which are conserved by time evolution. However, the invariance of the dynamics is defined by the invariance of the action. This is bigger than that of the Hamiltonian, because it also involves time-dependent transformations. For example, in a nonrelativistic theory the action must be invariant under Galilei boost: the change between two frames that move at constant velocity with respect to each other. But the Hamiltonian in general doesn't possess this invariance: Galilei boosts obviously change the values of the momenta, and the Hamiltonian in general depends on them. The set of operators which commute with all transformations that leave the action invariant defines the quantum numbers carried by elementary excitations of the system (elementary particles).

A nonrelativistic theory must have an action which is invariant upon the Galilei group. The Galilei transformations, along with the associate quantum-mechanical operators are [2]:

- space translations: $x_i \rightarrow x'_i = x_i + a_i$; $P_i = -i\partial_i$
- time translation: $t \rightarrow t' = t + a$; $H = i\frac{d}{dt}$
- Galilei boosts: $x_i \rightarrow x'_i = x_i + v_i t$; $p_i \rightarrow p_i + m v_i$; $K_i = -it\partial_i - mx_i$
- rotations: $x_i \rightarrow x'_i = R_{ij}x_j$; $J_i = \epsilon_{ijk}x^j\partial_k + \sigma_i$

The generator of rotations is the sum of orbital angular momentum and *spin*.

The generators of the Galilei group form the Galilei algebra:

$$\begin{aligned} [J_i, J_j] &= \epsilon_{ijk}J_k; & [P_i, P_j] &= 0; & [K_i, K_j] &= 0; & [J_i, H] &= [K_i, H] = 0; \\ [k_i, H] &= iP_i; & [J_i, P_j] &= \epsilon_{ijk}P_k; & [J_i, K_j] &= \epsilon_{ijk}K_k; & [K_i, P_j] &= iM\delta_{ij} \end{aligned} \quad (1)$$

In order to close the algebra it is necessary to introduce a (trivial) mass operator M which commutes with everything else:

$$[M, P_i] = [M, K_i] = [M, J_i] = [M, H] = 0. \quad (2)$$

The Casimir operators, which commute with all generators, are

$$C_1 = M; \quad C_2 = 2MP_0 - P_iP_i; \quad C_3 = (MJ_i - \epsilon_{ijk}P_jK_k)(MJ_i - \epsilon_{ilm}P_lK_m). \quad (3)$$

In terms of quantum-mechanical operators the Casimirs correspond to

- $C_1 = m$ (mass);

- $\frac{1}{2m}C_2 = -i\frac{d}{dt} - \frac{p^2}{2m}$ (internal energy);
- $C_3 = \sigma_i \sigma_i$ (**spin**).

We see that spin is one of the three numbers which classify nonrelativistic elementary excitations, along with mass and internal energy.

In the relativistic case, the action is invariant under the Poincaré group. The transformations and associate operators are now:

- translations: $x_\mu \rightarrow x'_\mu = x_\mu + a_\mu$; $P_\mu = -i\partial_\mu$
- Lorentz transf.:
 $x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$; $p^\mu \rightarrow p'^\mu = \Lambda^\mu_\nu p^\nu$;
 $J^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu + \Sigma^{\mu\nu}$
rotations: $J_i = \frac{1}{2}\epsilon_{ijk}J^k = \epsilon_{ijk}x_j P_k + \sigma^i$
boosts: $K_i = J^{i0}$.

The Poincaré generators form the algebra

$$\begin{aligned} [J^{\mu\nu}, J^{\rho\sigma}] &= i(g^{\mu\rho}J^{\nu\sigma} - g^{\nu\rho}J^{\mu\sigma} + g^{\mu\sigma}J^{\nu\rho} + g^{\nu\sigma}J^{\mu\rho}); \\ [P^\mu, J^{\rho\sigma}] &= -i(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho); \quad [K^\mu, P^\nu] = 0 \end{aligned} \quad (4)$$

Explicitly, the algebra of boosts and rotations is

$$\begin{aligned} [J_i, J_j] &= \epsilon_{ijk}J_k; \quad [J_i, K_j] = \epsilon_{ijk}K_k; \quad [K_i, K_j] = -i\epsilon_{ijk}K_k \\ [J_i, P_j] &= \epsilon_{ijk}P_k; \quad [K_i, H] = iP_i; \quad [K_i, P_j] = iH\delta_{ij}. \end{aligned} \quad (5)$$

The Casimir operators are now just two:

$$C_1 = P_\mu P^\mu; \quad C_2 = W_\mu W^\mu, \quad (6)$$

in terms of the momentum generator and the *Pauli-Lubanski* operator

$$W^\mu = \epsilon^{\mu\nu\rho\sigma}P_\nu J_{\rho\sigma}. \quad (7)$$

The corresponding quantum-mechanical operators are

- $C_1 = P^2$; eigenvalue M^2 (mass);
- $C_2 = W^2 = m\sigma^2$; eigenvalue $M^2s(s+1)$ (mass×**spin**),

where the latter identification is clear if one chooses the rest frame, as we shall discuss in greater detail in section 4.3.

Galilei transformations can be obtained from Poincaré transformations in the limit $v \ll 1$ by assuming the scaling laws $M \sim 1$, $J \sim 1$, $P \sim v$, $H \sim v^2$, $K \sim 1/v$.

Summarizing, both in nonrelativistic and relativistic theories spin is one of the quantum numbers that classify elementary excitations. In quantum mechanics, the state vectors of physical systems are expanded on a basis of irreducible representations of the rotation group (in the nonrelativistic

case) or the Lorentz group (in the relativistic case). In quantum field theory, one-particle states are, respectively, Galilei or Poincaré irreducible representations. In the relativistic case, rotations are implicitly defined by the Pauli-Lubanski vector eq. (7) as the subgroup of the Lorentz group which leaves the four-momentum invariant.

In more than two spatial dimensions the rotation group $O(d)$ is doubly connected (i.e., $\pi_1[O(d)] = \mathbb{Z}_2$); its universal cover is the group $\text{Spin}(d)$, which, in the usual $d = 3$ case, is isomorphic to $\text{SU}(2)$. When $d = 2$ (planar systems) the rotation group is $O(2)$, which, being isomorphic to the circle S^1 is infinitely connected ($\pi_1[O(2)] = \mathbb{Z}$); its universal cover is the real line \mathbb{R} . It follows that in more than two dimension the wave function can carry either a simple-valued (Bosons) or a double valued (Fermions) representation of the rotation group, and in two dimensions it may carry an arbitrarily multivalued one (anyons [4]).

The multivaluedness of the representation of rotations is is classified by the value of the phase which the wave function acquires upon rotation by 2π about an arbitrary axis (the z axis, say):

$$R_z^{2\pi} \psi(q_1, \dots, q_n) = e^{2\pi i J_z} \psi(q_1, \dots, q_n) = e^{2\pi i \sigma} \psi(q_1, \dots, q_n), \quad (8)$$

where $J_z = L_z + \sigma$, and in the last step we have used the fact that the spectrum of orbital angular momentum is given by the integers, so upon 2π rotation it is only spin that contributes to the phase.

2.2 Statistics and Topology

The wave function for a system of n identical particles must be invariant in modulus, and thus acquire a phase, upon interchange of the full set of quantum numbers q_i of the i -th and j -th particle:

$$\psi(q_1, \dots, q_i, \dots, q_j, \dots, q_n) = e^{2\pi i \sigma} \psi(q_1, \dots, q_j, \dots, q_i, \dots, q_n). \quad (9)$$

The parameter σ , which is only defined modulo integers, is the statistics of particles i, j . We now prove the spin-statistics theorem, which states that the statistics is a universal property of particles i, j , and it is equal to their spin (also in $d=2$, where the spin as we have seen can be generic).

The proof is based on an analysis of the quantisation of systems defined on topologically nontrivial configuration spaces. Indeed, if \mathcal{C}_d is the configuration space for a single particle in d dimensions, the configuration space for a system of n particles in d dimensions is

$$\bar{\mathcal{C}}_d^n = \mathcal{C}_d^n - \mathcal{D}, \quad (10)$$

where \mathcal{D} is the set of points where the full set of quantum numbers of two or more particles coincide. These points must be excised from space, otherwise eq. (9) with $x_i = x_j$ implies that necessarily $\sigma = 0$.

If the particles are identical, points which differ by their interchange must be identified. The configuration space then becomes the coset space

$$\mathcal{C}_d^n = \frac{\bar{\mathcal{C}}_d^n}{S_n}, \quad (11)$$

where S_n is the group of permutations of n objects. The topological structure of the configuration space changes when going from two to more than two dimensions, just like the topological structure of the rotation group discussed in section 2.2. Indeed, if $d = 2$ the space eq. (10), i.e. before dividing out permutations, is multiply connected: a closed path traversed by the i -th particle in which particle j is inside the loop formed by particle i cannot be deformed into a path in which particle j is outside the loop. The configuration space \mathcal{C}_2^n is then also multiply connected, and its fundamental group is the braid group $\pi_1(\mathcal{C}_2^n) = B_n$, as we shall discuss explicitly below.

In more than two dimensions, the space $\bar{\mathcal{C}}_d^n$ is simply connected: all closed path traversed by a particle can be continuously deformed into each other, because in more than two dimensions one cannot distinguish the inside of a one-dimensional curve from its outside. However, the configuration space \mathcal{C}_d^n is multiply connected. This implies that a topologically nontrivial closed path in \mathcal{C}_d^n must correspond to an open path in $\bar{\mathcal{C}}_d^n$, because all closed paths in $\bar{\mathcal{C}}_d^n$ can be deformed into each other. Furthermore, points in \mathcal{C}_d^n that correspond to same point in $\bar{\mathcal{C}}_d^n$ are in one-to-one correspondence with elements of S_n , because S_n acts effectively, i.e. only the identity of S_n maps all points of $\bar{\mathcal{C}}_d^n$ onto themselves. It follows that equivalence classes of paths in $\bar{\mathcal{C}}_d^n$ are in one-to-one correspondence with elements of the permutation group:

$$\pi_1(\mathcal{C}_d^n) = S_n. \quad (12)$$

Hence, the multiply connected nature of the configuration space is directly linked with the presence of identical particle, and specifically to the response of the system upon permutations, i.e. to statistics.

Therefore, let us consider quantization on a multiply connected space. The way nontrivial statistics is obtained can be understood by studying this problem in a path-integral approach [3], where transition amplitudes are written in terms of the propagator $K(q', q)$

$$S_{fi} \equiv \langle \psi_f | \psi_i \rangle = \langle \psi_f | q' t' \rangle \langle q' t' | q t \rangle \langle q t | \psi_i \rangle = \int dq dq' \psi_f^*(q') K(q', q) \psi_i(q), \quad (13)$$

which in turn can be written as a sum over paths

$$K(q', t'; q, t) = \int_{q(t)=q; q(t')=q'} Dq(t_0) e^{i \int_t^{t'} dt_0 L[q(t_0)]}. \quad (14)$$

Closed paths on a multiply connected space fall into homotopy classes. Moreover, open paths can also be classified in homotopy classes by a choice

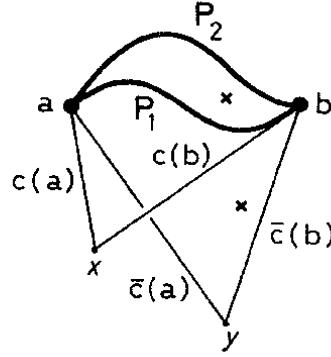


Fig. 1. Paths P_i are assigned to homotopy classes by connecting them to a base point through a mesh. Changing the base point from x to y can change the absolute class assignment of a path, but not the relative assignment of a pair of paths.

of *mesh* (figure 1). Namely, one chooses an arbitrary reference point x (base point) and then one assign to each point in space a path connecting it to the base point. The homotopy class of an open path can then be defined as the homotopy class of the closed path formed by the given open path and the mesh that connects it to the base point. Once all paths (closed and open) are grouped into equivalence classes, the path integral is in general defined as follows

$$K(q', t'; q, t) = \sum_{\alpha} \chi(\alpha) K^{\alpha}(q', t'; q, t), \quad (15)$$

where $K^{\alpha}(q', t'; q, t)$ is computed including in the sum over paths only paths in the α -th homotopy class, and $\chi(\alpha)$ are weights which depend only on the equivalence class (homotopy class) of a given path.

The weighted sum eq. (15) must satisfy the following physical requirements:

- (a) physical result must be independent of the choice of mesh;
- (b) amplitudes must satisfy the superposition principle, which in turn implies the convolutive property

$$K(q'', t''; q, t) = \int dq' \langle q'' t'' | q' t' \rangle \langle q' t' | q t \rangle = \int dq' K(q'', t'' q' t';) K(q' t'; q, t). \quad (16)$$

The necessary and sufficient condition for these requirements to be satisfied is that the weights $\chi(\alpha)$ satisfy

$$|\chi(\alpha)| = 1 \quad (17)$$

$$\chi(\alpha \circ \beta) = \chi(\alpha)\chi(\beta), \quad (18)$$

where in eq. (18) α and β are the homotopy classes of paths with a common endpoint, and $\alpha \circ \beta$ is the homotopy class of the path obtained by joining them.

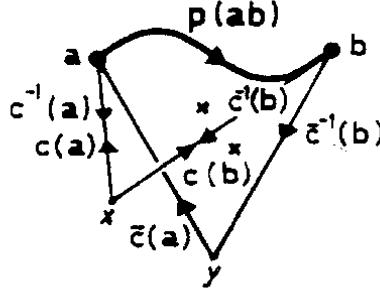


Fig. 2. Graphical representation of eq. (20)

The proof that eq. (18) implies property (b) is immediate:

$$\sum_{\gamma} \chi(\gamma) K^{\gamma}(q', t'; q, t) = \sum_{\alpha, \beta} \chi(\alpha) \chi(\beta) \int dq' K^{\alpha}(q'', t'' q' t'; q, t) K^{\beta}(q' t'; q, t). \quad (19)$$

The proof that eq. (17) implies property (a) is also easy: let P be the closed path obtained composing the open path p which connects points a and b with a mesh C (figure 2)). Upon changing the mesh to \bar{C} , the path P becomes the path \bar{P} , which in turn can be obtained by composing P with the closed paths $\lambda \equiv \bar{C}(a)C^{-1}(a)$ and $\mu = C(b)\bar{C}^{-1}(b)$:

$$\begin{aligned} \bar{P}(ab) &= \bar{C}(a)p(ab)\bar{C}^{-1}(b) \\ &= \bar{C}(a)C^{-1}(a)C(a)p(ab)C^{-1}(b)C(b)\bar{C}^{-1}(b) \\ &= \lambda P(ab)\mu. \end{aligned} \quad (20)$$

Because μ and λ do not depend on the original path, but only on the two meshes, the factor $\chi(\lambda\mu)$ which relates the two class assignments

$$\bar{\chi}(\alpha) = \chi(\lambda\mu)\chi(\alpha) \quad (21)$$

is universal. It follows that

$$\sum_{\gamma} \bar{\chi}(\gamma) K^{\alpha}(q', t'; q, t) = \chi(\lambda\mu) \sum_{\gamma} \chi(\gamma) K^{\alpha}(q', t'; q, t), \quad (22)$$

so if χ are phases the transition probability is unchanged.

This proves that conditions (17-18) are sufficient for requirements (a-b) to be satisfied, the proof that they are also necessary is somewhat more technical and we shall omit it. Conditions eq. (17-18), taken jointly, mean that phases χ provide one-dimensional unitary representation of $\pi_1(\mathcal{C}_d^n)$, i.e. the permutation group S_n (in more than two dimensions) or the braid group (in two dimensions).

2.3 Bosons, Fermions and Anyons

The relation between spin and statistics now follows from the structure of the path integral. First, we observe that there are only two unitary one-dimensional irreducible representations of the permutation group S_n : the trivial one (where $\chi = 1$ for all permutations), and the alternating one, where $\chi = 1$ if the permutation is even and $\chi = -1$ if it is odd (i.e., if they may be performed by an even or odd number of interchanges, respectively). Now, note that the wave function at time t is given by the path integral in terms of some boundary condition at time t_0 :

$$\langle q, t | \psi \rangle = \int dq_0 K(q, t; q_0, t_0) \psi_0(q_0, t_0). \quad (23)$$

Two evolutions that lead to final states which only differ by the interchange of the coordinates q_i, q_j in configuration space differ by the factor χ : hence, $\chi = -1$ correspond to $\sigma = \frac{1}{2}$ ($\sigma = 0$). However, an interchange of coordinates q_i, q_j can also be realized by a rotation by π of the system about any axis through the center of mass of the two particles (or a rotation about any axis followed by a translation), which in turn is generated by the corresponding angular momentum operator

$$|q_j q_i\rangle = e^{i\pi J_z^{ij}} |q_i q_j\rangle, \quad (24)$$

where J_z^{ij} is the component along the (arbitrarily defined) z axis of the angular momentum of particles i, j . The constraint that σ can only be either integer or half-integer is understood as a consequence of the trivial fact that two interchanges, or a rotation by 2π , must bring back to the starting configuration.

It follows that if $\chi = -1$, so $\sigma = \frac{1}{2}$, and the spectrum of J_z^{ij} is given by the odd integers. We can then view the contribution of χ to the path integral as the result of having added an extra internal effective interaction, which shifts the angular momentum of the pair of particles i, j by an integer, i.e. the angular momentum of each particle by a half-integer. This establishes the spin-statistics theorem in a nonrelativistic theory. The results is a consequence of the fact that fermionic statistics, which is usually given as a property of wave functions, has been lifted through the path-integral formalism to a property of particle paths, and attributed to a weight given to paths.

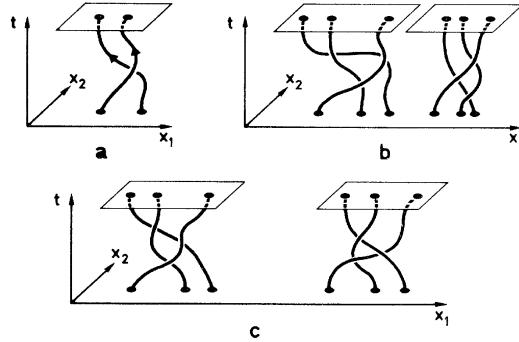


Fig. 3. Braids defined by particles' trajectories

The fact that either trivial or alternating representations of permutations are possible is then directly related to the existence of either single-valued or double-valued representations of rotations.

In order to understand this better, let us now consider the case of planar systems [4], both because we can then generalize this spin-statistics connection to arbitrary spin and statistics (anyons), and also because we can then work out an explicit representation for the effective interaction associated to the χ weights, which will lead us to the spin action which we shall then discuss in the next section.

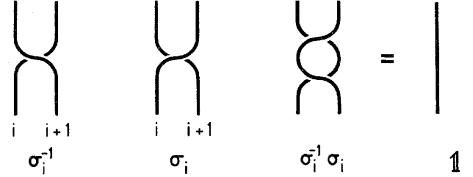


Fig. 4. The exchange operator σ_i and its inverse

In $d = 2$, χ^α provide an abelian irreducible representation of the braid group. Indeed, each particle trajectory on a multiply-connected space defines an inequivalent braid (figure 3). Each braid, in turn, is uniquely defined as a sequence of interchanges of pairs of neighbouring particles. This can be represented algebraically by introducing the operator σ_i which exchanges particles i and $i + 1$ (figure 4). Two braids are equivalent if they can be deformed into each other. For instance (figure 5)

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad (25)$$

$$\sigma_i \sigma_{i+1} \sigma_i \neq \sigma_i \sigma_{i+1} \sigma_i^{-1}. \quad (26)$$

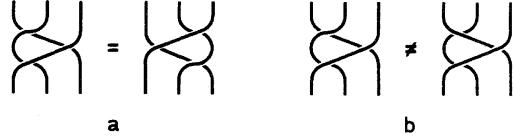


Fig. 5. Graphical representation of eq. (25) (a) and eq. (26) (b)

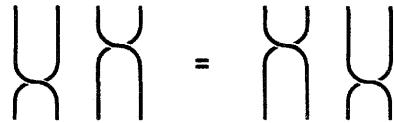


Fig. 6. Graphical representation of eq. (27)

In fact, all independent relations between braids are eq. (25) and

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| > 1. \quad (27)$$

In terms of χ eq. (27) implies

$$\chi(\sigma_i \sigma_j) = \chi(\sigma_i) \chi(\sigma_j) \quad \text{if } |i - j| > 1, \quad (28)$$

while eq. (25) implies

$$\chi(\sigma_i) = \chi(\sigma_j) \quad \text{for all } i, j. \quad (29)$$

Equations (28,29) in turn imply that the weight for a generic path (braid) is

$$\chi(\sigma_{i_1} \dots \sigma_{i_n}) = \chi(\sigma_{i_1}) \dots \chi(\sigma_{i_n}) = \exp \left(2i\sigma \sum_{k=1}^n \epsilon_k \right), \quad (30)$$

where $\epsilon = +1$ for a direct exchange and $\epsilon = -1$ for its inverse σ_i^{-1} , and σ coincides with the statistics parameter eq. (9). The cases of bosons and fermions are recovered when $\sigma = 0$ or $\sigma = \frac{1}{2}$, respectively, but now σ can take any real value (anyons). Indeed, in two dimensions two subsequent interchanges do not necessarily take back to the starting point, because a path where particle i traverses a loop encircling particle j cannot be shrunk to a

point (identity). Hence, two interchanges do not necessarily bring back to the starting configuration, and the constraint that $2\sigma = 1$ no longer applies. Accordingly, as already mentioned, in two dimension the rotation group admits arbitrarily multivalued representations.

The χ weights can be represented explicitly in terms of the variation of relative polar angle $\Theta(\mathbf{x}) \equiv \tan^{-1}\left(\frac{x^2}{x^1}\right)$ of particles i and j along the particles' paths:

$$\chi = \exp\left(-2i\sigma \sum_{i < j} \Delta\Theta_{ij}\right) = \exp\left(-i\sigma \sum_{i \neq j} \int dt \frac{d}{dt} \Theta(\mathbf{x}_i(t) - \mathbf{x}_j(t))\right). \quad (31)$$

Using this representation, the weighted path integral eq. (15) becomes

$$\begin{aligned} K(q', t'; q, t) &= \int_{q(t)=q; q(t')=q'} Dq(t_0) e^{i \int_t^{t'} dt_0 (L[q(t_0)] - \sigma \sum_{i \neq j} \frac{d}{dt_0} \Theta[\mathbf{x}_i(t_0) - \mathbf{x}_j(t_0)])} \\ &= \sum_{n_{ij}, (i \neq j) = -\infty}^{\infty} e^{-i\sigma (\sum_{i \neq j} \hat{\Theta}(\mathbf{x}_i(t') - \mathbf{x}_j(t')) + 2\pi n_{ij})} K_0^{(n)}(q', t'; q, t) e^{i\sigma \sum_{i \neq j} \hat{\Theta}(\mathbf{x}_i(t) - \mathbf{x}_j(t))}. \end{aligned} \quad (32)$$

Hence, the weights χ can be viewed as the consequence of having added to the Lagrangian L the effective interaction term

$$L_{\text{eff}}[q(t)] = -\sigma \sum_{i \neq j} \frac{d}{dt} \Theta[\mathbf{x}_i(t) - \mathbf{x}_j(t)]. \quad (33)$$

If the starting Lagrangian L described bosonic excitations, the interaction eq. (33) endows these excitations with statistics σ .

Equation (32) shows that the effect of the statistics-changing interaction can be absorbed in a redefinition of the wave function by a phase:

$$\psi_0(q, t) = e^{i\sigma \sum_{i \neq j} \Theta_{ij}(t)} \psi(q, t) : \quad (34)$$

the wave function ψ_0 is propagated by the path-integral defined in terms of the bosonic Lagrangian L . However, it is defined on a space of paths rather than a space of coordinates, and it satisfies “twisted” boundary conditions: upon rotation by 2π it acquires a phase

$$R^{2\pi} \psi_0(q, t) = e^{i2\pi\sigma n(n-1)} \psi_0(q, t), \quad (35)$$

and correspondingly the spectrum of eigenvalue of the angular momentum operator (which generates rotations) is

$$j = j_0 + \sigma n(n-1), \quad (36)$$

where j_0 is the spectrum of angular momentum for the original Lagrangian. We see explicitly that for a system of particles the angular momentum of the pair is shifted by 2σ i.e. each particle has acquired spin σ .

The effective statistics-changing Lagrangian L_{eff} eq. (33) looks intrinsically nonrelativistic, in that it depends on the polar angle as a function of time. However, it also admits a covariant formulation, which will turn out to be closely related to the formulation of a path integral for spin. To see this, define a covariant particle current

$$j^\mu = \sum_{i=1}^n \left(1, \frac{d\mathbf{x}_i}{dt} \right) \delta^{(2)}(\mathbf{x} - \mathbf{x}_i) = \sum_{i=1}^n \int ds \delta^{(3)}(x - x_i) \frac{dx^\mu}{ds}, \quad (37)$$

where s is any covariant parametrization along the particle path (e.g. the path-length). Furthermore, add to the action $I_0 = \int dt L(t)$ a covariant coupling of the current to a gauge potential A_μ :

$$I = I_0 + I_c + I_f \quad (38)$$

$$I_c = \int d^3x j^\mu(x) A_\mu(x) \quad (39)$$

$$I_f = -\frac{1}{2\sigma} \int d^3x \epsilon^{\mu\nu\rho} A_\mu(x) \partial_\nu A_\rho(x). \quad (40)$$

The action I_c for the gauge potential A_μ is quadratic and can be integrated out:

$$I_{\text{eff}}[j] \equiv -i \ln \int \mathcal{D}A^\mu e^{i(I_c + I_f)} = \pi\sigma \int d^3x d^3y j^\mu(x) K_{\mu\nu}(x, y) j^\nu(y), \quad (41)$$

in terms of the Green function $K_{\mu\nu}(x, y)$ for the operator $\epsilon_{\mu\rho\nu} \partial_\nu$:

$$K_{\mu\nu}(x, y) = -\frac{1}{2\pi} \epsilon_{\mu\rho\nu} \frac{(x-y)^\rho}{|x-y|^3} \quad (42)$$

$$\epsilon_{\mu\nu\rho} \partial_\nu K^{\rho\sigma}(x, y) = \delta_\mu^\sigma \delta^{(3)}(x-y). \quad (43)$$

The effective current-current interaction

$$I_{\text{eff}} = \sigma \sum_{i,j} I_{ij}, \quad I_{ij} = -\frac{1}{2} \int dx_i^\mu dx_j^\nu \epsilon_{\mu\rho\nu} \frac{(x_i - x_j)^\rho}{|x_i - x_j|^3} \quad (44)$$

is formally identical to the interaction of the current j^μ with a Dirac magnetic monopole potential \tilde{A}_μ :

$$\frac{x^\mu}{|x|^3} = \epsilon^{\mu\alpha\beta} \partial_\alpha \tilde{A}_\beta(x). \quad (45)$$

It is now easy to recover the form eq. (33) of the spin-statistics changing interaction. To this purpose, we choose an explicit ‘‘Coulomb gauge’’ representation for the potential $\tilde{A}_\beta(x)$:

$$\tilde{A}_\mu(t, \mathbf{x}) = \left(0, -\frac{\epsilon_{ab} x^b}{r(t-r)} \right), \quad r^2 = |x|^2 = t^2 - x_1^2 - x_2^2, \quad (46)$$

and we parametrize paths with time, $s = t$.

We get

$$\begin{aligned} I_{ij} &= -\frac{1}{2} \int_0^T dt \int_0^T dt' \frac{dx_i^\mu(t)}{dt} \left(\partial_\mu \tilde{A}_\nu(x_i - x_j) - \partial_\nu \tilde{A}_\mu(x_i - x_j) \right) \frac{dx_j^\nu(t')}{dt'} \\ &= \int_0^T dt \epsilon^{ab} \left(\frac{dx_i^a}{dt} - \frac{dx_j^a}{dt} \right) \frac{(x_i(t) - x_j(t))^b}{|x_i(t) - x_j(t)|^2} + I_g, \end{aligned} \quad (47)$$

where I_g is a rotationally invariant surface term which has no effect on spin and statistics. Now, terms with $i = j$ in eq. (47) vanish by antisymmetry, while terms with $i \neq j$ can be rewritten using the identity

$$\partial_a \Theta(\mathbf{x}) = -\epsilon^{ab} \frac{x^b}{|\mathbf{x}|^2}, \quad (48)$$

which immediately implies that

$$I_{ij} = - \int dt \frac{d}{dt} \Theta(\mathbf{x}_i - \mathbf{x}_j) + I_g, \quad (49)$$

i.e., up to the irrelevant I_g , the same as the action obtained from the effective Lagrangian eq. (33).

Summarizing, we have found that nontrivial statistics is enforced by weighing topologically inequivalent paths in the path integral, that inequivalent paths are those which correspond to interchanging the coordinates of two or more particles, and that these weights can be obtained as the result of adding to the Lagrangian an effective interaction term, which shifts the spectrum of the total angular momentum by a half-integer contribution per particle. Furthermore, in two dimensions we have obtained an explicit local representation of this effective interaction term, which is formally equivalent to the interaction of the particle current with a Dirac magnetic monopole localized on each other particle.

3 A Path Integral for Spin

Spin is usually quantized by introducing degrees of freedom which live in an internal space. In particular, the quantization of Fermions is usually performed by introducing anticommuting variables. However, in the previous section we have seen that it is possible to represent the effect of fermionic statistics in terms of an interaction defined in configuration space, and then path-integrating over this space. In this section we shall see that it is also possible to obtain the path-integral quantization of a spin degree of freedom by constructing the configuration space for a classical spin, and then path-integrating over evolutions in this configuration spaces with a suitable weight.

3.1 The Spin Action

It is well-known that the classical action for a free (relativistic) particle coincides with the arc-length L of the path $x^\mu(s)$ traversed by it, and in fact its quantization [12] can be obtained by summing over paths with a weight given by an action which coincides with the arc-length L :

$$I = m \int ds \sqrt{\left(\frac{dx^\mu}{ds}\right)^2} = mL. \quad (50)$$

Hence, the quantization of the spinning particle is obtained by first defining the space of paths, and then introducing as a weight over it the simplest geometric invariant of the paths.

The path-integral quantization of spin can be obtained in a similar way. The configuration space for spin is the set of points spanned by a vector

$$\mathbf{s} = \sigma \mathbf{e} \quad (51)$$

with fixed modulus $|\mathbf{s}| = \sigma$, namely the two-sphere S^2 . This can be viewed as the result of the action of the rotation group on a reference vector, namely, the coset of the rotation group over the subgroup of rotations that leave the reference vector invariant (little group): $S^2 = SO(3)/SO(2)$. The simplest invariant over this space is the solid angle subtended by a closed path. Therefore, parametrizing the vector \mathbf{e} in spherical coordinates

$$\mathbf{e} = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix} \quad (52)$$

we define the spin action as

$$I_s = \int dt \mathcal{L}(\theta, \phi) = s \int dt \cos \theta \dot{\phi}. \quad (53)$$

Equation (53) provides us with a spin action in the sense that the time-evolution (transition amplitude) for spin wave functions

$$|\phi\rangle = |m\rangle \langle m|\phi\rangle; \quad \langle m|\phi\rangle = \frac{e^{-im\phi}}{\sqrt{2\pi}} \quad (54)$$

is given by

$$\langle f | i \rangle = \langle \phi_f | e^{i \int H(t) dt} | \phi_i \rangle, = \int_{\mathbf{e}(t_f) = \mathbf{e}(\phi_f); \mathbf{e}(t_i) = \mathbf{e}(\phi_i)} D\mathbf{e} e^{i \int dt \mathcal{L}_s - V(\mathbf{J})}, \quad (55)$$

where $H(t)$ is a Hamiltonian which describes the spin dynamics (e.g. the coupling with an external magnetic field, $H = \mathbf{s} \cdot \mathbf{B}$) and the boundary

conditions are given in terms of ϕ only (which is equivalent to specifying an eigenvalue m of the third component of angular momentum). This result can be proven by direct computation [5, 6]. We shall instead first show that the action I_s eq. (53) leads to the correct classical dynamics of spin, then quantize it using the general principles of geometric quantization.

Let us first take a closer look at the spin action. Its geometric interpretation becomes apparent by rewriting it as

$$I_s = \sigma \int_C \cos \theta \dot{\phi} dt = \sigma \int_C \cos \theta d\phi \quad (56)$$

$$= \sigma \int_S d\cos \theta d\phi = \sigma \int_S d\mathbf{S} \cdot \mathbf{e} = \sigma \int_S \left(\frac{\partial \mathbf{e}}{\partial s} \times \frac{\partial \mathbf{e}}{\partial t} \right) \cdot \mathbf{e}, \quad (57)$$

where C is the path traversed by the vector \mathbf{e} eq. (52), and the second step eq. (57) holds when the path is closed, in which case S is the surface bound by C . In such case, eq. (57) shows explicitly that the spin action coincides with the solid angle subtended by C

Equation (57) shows manifestly that there is a 4π ambiguity in the definition of the spin action, which corresponds to the possibility of choosing the upper or lower solid angle subtended by C on the sphere. In order for this ambiguity to be irrelevant, the parameter σ , which as we shall see corresponds to the value of spin, must be quantized in half-integer units. The connection between the spin action and the effective two-dimensional statistics action eq. (44) becomes clear by rewriting it as

$$I_s = \sigma \int_S d\mathbf{S} \cdot \nabla \times \tilde{\mathbf{A}}[\mathbf{e}], \quad (58)$$

where $\tilde{\mathbf{A}}$ is the Dirac monopole potential eq. (45), in the space of spin vectors:

$$\mathbf{e} = \nabla \times \tilde{\mathbf{A}}[\mathbf{e}] \quad (59)$$

3.2 Classical Dynamics

In order to verify that the spin action defines the action for a classical spin degree of freedom, we first check that it leads to the Poisson bracket

$$\{s^i, s^j\} = \epsilon^{ijk} s^k. \quad (60)$$

This is easily done using the Faddeev-Jackiw formalism [7] for the Hamiltonian treatment of systems defined by first-order Lagrangians, i.e. by a Lagrangian of the form

$$L = f_i(x) \frac{dx_i}{dt} - V(x). \quad (61)$$

Namely, it is easy to see that the Euler-Langrange equations implied by the Lagrangian eq. (61) have the form

$$f_{ij} \frac{dx_j}{dt} = \frac{\partial V}{\partial x_i} \quad (62)$$

$$f_{ij} \equiv \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}. \quad (63)$$

This coincides with the canonical Hamiltonian form

$$\frac{dx^i}{dt} = \{x^j, x^i\} \frac{\partial V}{\partial x^j} = \{V, x^i\} \quad (64)$$

if the Poisson brackets are given by

$$\{x^i, x^j\} = (f^{-1})^{ji} \quad (65)$$

It can be shown that the same result is found in the more standard approach, where the Lagrangian eq. (61) is viewed as defining a constrained dynamics, which is then treated defining suitable Dirac brackets.

Specializing this formalism to the spin action we see that its Dirac monopole form eq. (58) has the form of eq. (61) with

$$f_i = \sigma \tilde{A}_i[e]. \quad (66)$$

Using the definition eq. (63) this leads to

$$f_{ij} = \sigma \left(\partial_i \tilde{A}_j - \partial_j \tilde{A}_i \right) = \sigma \epsilon^{ijk} e^k. \quad (67)$$

Because

$$f_{ij}^{-1} = \frac{1}{\sigma^2} f_{ij} \quad (68)$$

the Poisson Brackets are

$$\{e^i, e^j\} = \frac{1}{\sigma} \epsilon^{ijk} e^k, \quad (69)$$

which, identifying the spin vectore \mathbf{s} with

$$\mathbf{s} = \sigma \mathbf{e}, \quad (70)$$

immediately lead to the spin Poisson brackets eq. (60). This also shows that the parameter σ gives the value of spin.

3.3 Geometric Quantization

The spin action can be quantized using the formalism of geometric or “coadjoint orbit” quantization [8, 9]. Namely, we view the time evolution of the (unit) spin vector $\mathbf{e}(t)$ as the result of the action of a rotation matrix $\Lambda(t)$ on a reference vector \mathbf{e}_0 :

$$\mathbf{e}(t) = \Lambda(t) \mathbf{e}_0. \quad (71)$$

This defines a path (orbit) in $S^2 = SO(3)/SO(2)$, where $SO(2)$ is the little group of \mathbf{e}_0 (the set of Λ matrices which leaves \mathbf{e}_0 invariant).

The path in S^2 can be lifted to a path in $SO(3)$ by assigning a frame, e.g. by defining the vector

$$\mathbf{n}(t) \equiv \frac{\dot{\mathbf{e}}(t)}{|\dot{\mathbf{e}}(t)|} \quad (72)$$

which satisfies

$$\mathbf{n} \cdot \mathbf{e} = 0. \quad (73)$$

The triple \mathbf{e} , \mathbf{n} , and

$$\mathbf{b}(t) \equiv \mathbf{e}(t) \times \mathbf{n}(t). \quad (74)$$

defines a time-dependent frame, which coincides with the standard Frenet frame if $\mathbf{e}(t)$ is viewed as the tangent vector to some path, in which case \mathbf{n} and \mathbf{b} are the unit normal and binormal, respectively. The matrix Λ is then fully specified by eq. (71) and

$$\mathbf{n}(t) = \Lambda(t)\mathbf{n}_0. \quad (75)$$

It is convenient in particular to choose the set of reference vectors

$$\begin{cases} \mathbf{v}^{(3)_0} = \mathbf{e}_0 \\ \mathbf{v}_0^{(1)} = \mathbf{n}_0 \\ \mathbf{v}_0^{(2)} = \mathbf{b}_0 \end{cases} \quad (76)$$

as

$$v_0^{(a)}{}_i = \delta_i^a. \quad (77)$$

It is then easy to see that the quantity

$$\left(\Lambda^{-1} \dot{\Lambda} \right)^{ij} = \mathbf{v}^{(i)} \cdot \dot{\mathbf{v}}^{(j)} \quad (78)$$

is an element of the $SO(3)$ algebra, the so-called Maurer-Cartan form, given by

$$\left(\Lambda^{-1} \dot{\Lambda} \right)_{ij} = \sum_{ab} C_{ab} (M^{ab})_{ij}; \quad (M^{ab})_{ij} = (\delta_i^a \delta_j^b - \delta_j^a \delta_i^b). \quad (79)$$

The coefficients C_{ij} can be extracted by exploiting the fact that the generators are orthogonal under tracing:

$$C_{ij} = \frac{1}{4} \text{tr} \left(M_{ij} \Lambda^{-1} \dot{\Lambda} \right) = \frac{1}{2} \mathbf{v}^{(i)} \cdot \dot{\mathbf{v}}^{(j)}. \quad (80)$$

We can now use this geometric formalism to rewrite yet again the spin action eq. (53) as

$$\begin{aligned} I_s &= \sigma \int_S \left(\frac{\partial \mathbf{e}}{\partial s} \times \frac{\partial \mathbf{e}}{\partial t} \right) \cdot \mathbf{e} = \sigma \int dt \, \dot{\mathbf{b}} \cdot \mathbf{n} + \text{integers} \\ &= \sigma \left(\text{tr} \int dt \frac{1}{2} \left(\Lambda^{-1} \dot{\Lambda} M_{12} \right) + \text{integers} \right). \end{aligned} \quad (81)$$

Note that any spin-dependent potential $V(\boldsymbol{\sigma})$ can be re-written in terms of Λ by exploiting eq. (80) to express the spin vector \mathbf{e} in terms of Λ :

$$e^i = \sigma \epsilon^{ijk} \left(\Lambda^{-1} \frac{M_{12}}{2} \Lambda \right)_{jk}. \quad (82)$$

This new form eq. (81) of the spin action has a twofold advantage: first, it does not depend on the representation, and second, it is amenable to geometric quantization. To demonstrate its representation-independence, let us show how the spinor representation is recovered from it. For spin $\frac{1}{2}$, the generators are

$$M_{ij} = -i\epsilon^{ijk}\sigma_k, \quad (83)$$

where σ_i are the usual Pauli matrices. We then have

$$\text{tr} \frac{1}{2} \left(\Lambda^{-1} \dot{\Lambda} M_{12} \right) = \text{tr} \left(\Lambda^{-1} \dot{\Lambda} \frac{\sigma_3}{2i} \right) = \text{tr} \left(\Lambda^{-1} \dot{\Lambda} \left(\frac{\mathbb{1} + \sigma_3}{2i} \right) \right). \quad (84)$$

The connection to (Pauli) spinors is found by introducing the reference two-component spinor

$$\psi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (85)$$

upon which the matrix Λ is taken to act in the spinor representation, namely

$$\psi(t) = T[\Lambda(t)]\psi_0, \quad (86)$$

where $T[\Lambda(t)]$ is the spinor representation of the rotation Λ . The relation between the spinor and vector representation is provided by constructing spinor bilinears

$$\psi^* \sigma^i \psi = \Lambda_{ij} \psi_0^* \sigma^i \psi_0 = \Lambda_{ij} e_0^j. \quad (87)$$

Using this relation, and noting that

$$|\psi_0\rangle\langle\psi_0| = \left(\frac{\mathbb{1} + \sigma_3}{2} \right) \quad (88)$$

it is easy to rewrite the spin action I_s eq. (81) as

$$I_s = \frac{1}{2} \int \frac{dt}{i} \psi^*(t) \frac{d}{dt} \psi(t), \quad (89)$$

which has the form of the kinetic term for a Pauli spinor $\psi(t)$. A generic spin-dependent potential $V(\boldsymbol{\sigma})$ can be written in terms of ψ by using the relation

$$\mathbf{e} = \psi^*(t) \boldsymbol{\sigma} \psi(t). \quad (90)$$

The quantization of spin is now reduced to the general problem [9] of quantizing a system whose configuration space is the space of states $|\psi\rangle$ which are orbits of a group G :

$$|\phi\rangle = T(g)|\phi_0\rangle, \quad (91)$$

where $g \in G$ is an element of the group of which $T(g)$ provides a unitary representation. The axioms of quantum mechanics imply that transition amplitudes for this system are given by the path integral

$$\langle f|i\rangle = \int Dg e^{iI_w[g]} \quad (92)$$

with the action

$$I_w[g] = \int dt \langle \phi_0 | \left[T(g^{-1}(t)) \frac{d}{dt} T(g(t)) - H(g(t)) \right] |\phi_0\rangle, \quad (93)$$

where H is a generic spin-dependent potential (or Hamiltonian, which coincides with the potential for a first-order Lagrangian).

The spin action eq. (81) is seen to coincide with the kinetic term of the geometric action eq. (93) if one identifies the representation matrix $T(g)$ with Λ eq. (71), and one observes that the projector on the state $|\phi_0\rangle$ can be expressed in terms of the generator $C_{ij}^0 M^{ij}$ of the little group of $|\phi_0\rangle$:

$$|\phi_0\rangle\langle\phi_0| = C_{ij}^0 M^{ij}. \quad (94)$$

Indeed, we get

$$\int dt \langle \phi_0 | \left[T(g^{-1}(t)) \frac{d}{dt} T(g(t)) \right] |\phi_0\rangle = \int dt \text{tr} C_{ij}^0 M^{ij} \Lambda^{-1}(t) \dot{\Lambda}(t) \quad (95)$$

which coincides with the spin action if we choose $C_{ij}^0 M^{ij} = M_{12}$. Hence, the spin path integral eq. (55) with the spin action eq. (53) follows from geometric quantization of the space of $SO(3)$ orbits.

The relation of this result to the usual sum over paths à la Feynman is apparent if we specialize again to the case of spin $\frac{1}{2}$. The sum over paths is performed by dividing the time evolution from t_i to t_f into discrete time steps $\Delta t = \frac{t_f - t_i}{N}$ so that $t_j = t_i + (j - 1)\Delta t$, and then letting $N \rightarrow \infty$. For a spin system we get

$$\langle f|i\rangle = \langle \psi_f | e^{-i \int_{t_i}^{t_f} H dt} | \psi_i \rangle = \prod_{j=1}^N \int d\Lambda_j \langle \psi_{j+1} | e^{-i \Delta t H(t_j)} | \psi_j \rangle. \quad (96)$$

The evolution along an infinitesimal time slice is then given by

$$\begin{aligned} \langle \psi_{j+1} | e^{-i \Delta t H(t_i)} | \psi_j \rangle &\approx \langle \psi_{j+1} | (1 - i \Delta t H(t_j)) | \psi_j \rangle \\ &= 1 + \frac{1}{2} \Delta t \psi^* \frac{d}{dt} \psi - i \Delta t H(t_j) \\ &\approx e^{i[\psi^* \frac{d}{dt} \psi - \Delta t H(t_i)]}, \end{aligned} \quad (97)$$

which coincides with the geometric quantization result eq. (93). The first-order quantization of spin is a simple consequence of the fact that a spin Hamiltonian does not contain a quadratic kinetic term: the action is then entirely determined by the first-order parallel transport of the spin vector.

The meaning of these results is that first, the probability for the time evolution between two spin states is given by

$$\langle f | i \rangle = \int_{\mathbf{e}_f(t_f) = \mathbf{e}_f(\Lambda_f); \mathbf{e}_i(t_i) = \mathbf{e}_i(\Lambda_i)} D\mathbf{e} e^{i [I_s[\mathbf{e}] - \int dt H(t, \mathbf{e})]} \quad (98)$$

and furthermore, the matrix element of any spin-dependent operator $F(\boldsymbol{\sigma})$ can be determined as

$$\langle f | F(\boldsymbol{\sigma}) | i \rangle = \int_{\mathbf{e}_f(t_f) = \mathbf{e}_f(\Lambda_f); \mathbf{e}_i(t_i) = \mathbf{e}_i(\Lambda_i)} D\mathbf{e} e^{i [I_s[\mathbf{e}] - \int dt H(t, \mathbf{e})]} F(\mathbf{e}). \quad (99)$$

Summarizing, we have seen that the path-integral quantization of a “static” spin degree of freedom — as e.g. in the Heisenberg model — can be given in terms of a geometrically determined first-order spin action. The usual formalism in the spin- $\frac{1}{2}$ case is obtained by specializing to the spinor representation of spin vectors, but it does not require anticommuting variables or relativity. It is interesting to note that the same results can be obtained from the well-known “Schwinger boson” representation of angular momentum operators in terms of creation and annihilation operators for the (bosonic) harmonic oscillator [11], by quantizing the harmonic oscillator degrees of freedom in terms of a first-order action [9] (i.e. in terms of coherent states).

4 Relativistic Spinning Particles

As discussed in the introduction, in a relativistic theory physical states are irreducible representations of the Poincaré group, i.e. they carry mass and spin: the one-particle state $|m, s\rangle$ satisfies

$$P^2|m, s\rangle = m^2|m, s\rangle; \quad W^2|m, s\rangle = m^2 s(s+1), \quad (100)$$

where the Pauli-Lubanski operator W , defined in eq. (7), generates Lorentz transformations which leave the particle momentum invariant, because by construction $W_\mu P^\mu = 0$. In particular, in the rest frame of the particle (for massive particles) $p = (m, \mathbf{0})$, so $W = (0, \mathbf{s})$. In a general frame, spin spans the three dimensional ($d - 1$ dimensional) space orthogonal to momentum. This introduces a coupling between spin and momentum which determines the dynamics of a relativistic spinning particle, both at the classical and quantum level.

4.1 Path Integral for Spinless Particles

Before discussing the quantization of spinning particles, let us review the path-integral quantization of a massive spinless particle [12]. As we mentioned already, the action eq. (50) of a spinless free particle, or the kinetic term in the action for an interacting spinless particle, coincides with the arc-length of the path traversed by the particle. This can be written in various equivalent ways: the simple integral of the arc-length element $ds = \sqrt{dx^\mu dx_\mu}$ eq. (50) can be rewritten in terms of an induced metric $g(s)$ along the path

$$I_0 = \int ds \left[\frac{1}{\sqrt{g}}^{\frac{1}{2}} \left(\frac{dx^\mu}{ds} \right)^2 + \frac{m^2}{2} \sqrt{g} \right]. \quad (101)$$

Both at the classical and at the quantum level, the equation of motion for g is the constraint

$$g = \frac{\dot{x}^2}{m^2}, \quad (102)$$

which shows that indeed $g(s)$ is the induced metric

$$dx^2 = g(s)ds^2, \quad (103)$$

and leads back to the original form eq. (50) of the action when substituted in eq. (101).

The action eq. (101) can in turn be rewritten in first-order form

$$I_0 = \int ds \left[p_\mu \frac{dx^\mu}{dt} - \frac{\sqrt{g}}{2} (p^2 - m^2) \right], \quad (104)$$

where the momentum (tangent vector) p^μ is also fixed by a constraint

$$p^\mu = \frac{1}{\sqrt{g}} \dot{x}^\mu \quad (105)$$

which again leads back to the original form eq. (101) when substituted in the action eq. (104). This first-order form of the action is the most suitable for geometric quantization, i.e. for describing the dynamics of the spinning particle similarly to the way we have described the dynamics of spin in section 3.3. The classical equations of motion can be obtained from any of these equivalent forms of the action, and express energy-momentum conservation. For instance, using the first-order form eq. (104) we get immediately the Euler-Lagrange equations

$$\frac{d}{dt} p^\mu = 0, \quad p^2 = m^2. \quad (106)$$

Path-integral quantization [12] can be performed by exploiting the “gauge invariance”, i.e. the reparametrization invariance of the system [8]. The (Euclidean) path integral

$$\langle x' | x \rangle = \mathcal{N} \int_{x(0)=x; x(1)=x'} Dx(s) e^{-m \int_0^1 ds \sqrt{\dot{x}^2}} \quad (107)$$

can be rewritten introducing the induced metric $g(s)$ eq. (102) as

$$\langle x' | x \rangle = \mathcal{N} \int_{x(0)=x; x(1)=x'} Dx(s) Dg(s) \delta^{(\infty)}(\dot{x}^2 - g) e^{-m \int_0^1 ds \sqrt{g}}. \quad (108)$$

Reparametrization invariance is now manifest, because upon a general reparametrization $s \rightarrow f(s)$, the metric $g(s)$ transforms as $g(s) \rightarrow g(f(s))[\dot{f}(s)]^2$. We can now perform the path integral by fixing the gauge, e.g. by imposing the condition

$$\dot{g}(s) = 0. \quad (109)$$

Because the path-length is

$$L = \int_0^1 ds \sqrt{\dot{x}^2} = \int_0^1 ds \sqrt{g(s)} \quad (110)$$

the gauge condition (109) implies

$$g(s) = L^2. \quad (111)$$

We can thus write the gauge-fixed path-integral as

$$\begin{aligned} \langle x' | x \rangle &= \mathcal{N} \int_0^\infty dL \int_{x(0)=x; x(1)=x'} Dx(s) Dg(s) \delta^{(\infty)}(\dot{x}^2 - g) \delta(g - L^2) e^{-mL} \\ &= \mathcal{N} \int_0^\infty dL \int_{x(0)=x; x(1)=x'} Dx(s) \delta^{(\infty)}(\dot{x}^2 - L^2) e^{-mL}. \end{aligned} \quad (112)$$

After gauge-fixing, a residual integration over path lengths L remains.

The path-integral can be re-written in terms of geometric variables along the path: this leads to geometric quantization again. We introduce a tangent vector along the path, which for classical paths (those which satisfy the Euler-Lagrange equations) coincides with the particle four-momentum:

$$e^\mu = \frac{\dot{x}^\mu}{|\dot{x}|} = \frac{\dot{x}^\mu}{L}. \quad (113)$$

We can replace the path-integration over trajectories by a path-integration over the tangent vectors e^μ . However, the boundary conditions now become a non-local constraint:

$$x^{\mu'} - x^\mu = \int_0^L ds e^\mu(s). \quad (114)$$

We thus get finally

$$\begin{aligned}\langle x' | x \rangle &= \mathcal{N} \int_0^\infty dL \int De(s) e^{-mL} \delta^{(\infty)}(e^2 - 1) \delta^{(3)}(x^{\mu'} - x^\mu - \int_0^L ds e^\mu(s)) \\ &= \mathcal{N} \int dL d\mathbf{p} \int De(s) e^{-mL} \delta^{(\infty)}(e^2 - 1) e^{i\mathbf{p} \cdot (x' - x - \int_0^L ds e(s))}.\end{aligned}\quad (115)$$

The usual expression of the bosonic (Klein-Gordon) propagator is obtained by regularizing the formal expression eq. (115). To this purpose, we cut off paths which are coarse on a scale $\sim \epsilon$ (where, of course ϵ has the dimensions of [length]). We then take the continuum limit with a mass renormalization condition, expressed by defining a renormalized mass M_{phys} such that

$$m \propto \epsilon M_{\text{phys}}^2. \quad (116)$$

The propagator $K(p)$ is obtained as the Fourier transform of the renormalized position-space amplitude:

$$\begin{aligned}K(p) &= \lim_{\epsilon \rightarrow 0} \mathcal{N} \int dL e^{-mL} \int De(s) e^{-\frac{\epsilon}{2} \int_0^L ds e^2} e^{-ip \cdot \int_0^L ds e(s)} \delta^{(\infty)}(e^2 - 1) \\ &= \mathcal{N} \int dL e^{-L\epsilon M_{\text{phys}}^2} e^{-L\epsilon p^2} = \mathcal{N} \frac{1}{p^2 + M_{\text{phys}}^2}.\end{aligned}\quad (117)$$

Up to the irrelevant albeit infinite normalization constant \mathcal{N} , we have thus recovered the standard form of the Klein-Gordon propagator.

4.2 The Classical Spinning Particle

The spinning particle is now obtained by coupling a spin degree of freedom to the spinless particle of section 4.1, with dynamics governed by the action discussed in section 3.1. This can be done in an elegant geometric way by combining the translational and spin configuration spaces. To this purpose, in one time and $d - 1$ space dimensions, we define a set of $d - 1$ orthonormal vectors e^μ , $n_{(1)}^\mu, \dots, n_{(d-2)}^\mu$, which can in turn be obtained by action of a Lorentz transformation matrix Λ on a set of reference vectors

$$\begin{cases} e^\mu &= \Lambda^\mu_\nu \hat{t}^\nu \\ n_{(i)}^\mu &= \Lambda^\mu_\nu \hat{n}_{(i)}^\nu \end{cases} \quad (118)$$

The reference vectors

$$\hat{t}^\mu = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}, \quad \hat{n}_0^{(i)\mu} = \delta_i^\mu \quad (119)$$

define a basis in one time and $d - 1$ space dimensions. The set of vectors e^μ , $n_{(i)}^\mu$ completely specifies the matrix Λ : indeed, the first vector has $d - 1$ independent components (being unimodular), the second, orthogonal to it, has $d - 2$ independent components and so on, so that overall they have

$\sum_{i=0}^{d-2} (d-i-1) = \frac{1}{2}d(d-1)$ independent components, like the $O(d-1, 1)$ matrix Λ .

In the four-dimensional case we are interested in, the matrix Λ has six independent components. We take the vector e^μ as the unit tangent to the particle trajectory, so that classically it is identified with momentum up to an overall factor of m :

$$e^\mu = \frac{\dot{x}^\mu}{|\dot{x}|}; \quad p^\mu = m e^\mu \quad (120)$$

and at the quantum level it is the variable one path-integrates over (compare eq. (115)). The vector $n_{(1)}^\mu$ is then identified with the spin vector discussed in the previous section, it has two independent components and lives in the S^2 orthogonal to e^μ :

$$e_\mu n_{(1)}^\mu = 0; \quad s^\mu = \sigma n_{(1)}^\mu \quad (121)$$

At the quantum level, the two independent vectors p^μ and s^μ entirely specify the configuration of the system, whereas at the classical level the canonical coordinate x^μ must also be given.

The action for the spinning particle is now simply obtained by combining the action for the spinless particle eq. (104) with the spin action eq. (81): by writing both in terms of Λ , the momentum-spin orthogonality constraint is automatically enforced. We get

$$I = \int ds \left[p_\mu \frac{dx^\mu}{dt} - \frac{\sqrt{g}}{2} (p^2 - m^2) \right] + \sigma \text{tr} \left(\Lambda^{-1} \dot{\Lambda} M_{12} \right). \quad (122)$$

It is straightforward to check that, at the classical level, the correct dynamics is obtained: the Euler-Lagrange equations are found by varying the action upon the most general Poincaré transformation, namely a translation of x^μ , and a Lorentz transformation of Λ . The variation upon translations gives trivially the spinless equation of motion eq. (106) (energy-momentum conservation). The most general Lorentz variation is

$$\delta \Lambda = i \omega^{\mu\nu} M_{\mu\nu} \Lambda, \quad (123)$$

upon which the action transforms as

$$\delta I = -i \text{tr} (\omega^{\mu\nu} M_{\mu\nu} K) + i \sigma \text{tr} \left(S \frac{d}{dt} \omega^{\mu\nu} M_{\mu\nu} \right) \quad (124)$$

$$K_{\mu\nu} \equiv (\dot{x}_\mu p_\nu - x_\nu \dot{p}^\nu) \quad (125)$$

$$S_{\mu\nu} = \sigma (\Lambda^{-1} M_{12} \Lambda)_{\mu\nu}. \quad (126)$$

Demanding that the action be stationary leads to the Euler-Lagrange equations

$$\frac{d}{dt} (x^\mu p^\nu - x^\nu p^\mu + S^{\mu\nu}) = 0. \quad (127)$$

Equation (127) expresses the set of conservation laws of a Lorentz invariant Lagrangian: in particular, the (i, j) components give the conservation

of (total) angular momentum, while the $(0, i)$ components give the equation $\mathbf{p} = \frac{d}{dt}(\mathbf{x}E)$ which relates momentum to velocity in the usual way.

4.3 Quantum Spinning Particles and Fermions

The dynamics of the spinning particle, described by the action eq. (122), is given on the space of Lorentz orbits $\Lambda(t)$ which evolve according to eq. (118) the pair of vectors p^μ eq. (120), s^μ eq. (121). The path integral then follows from geometric quantization [9, 10] eqs. (92,93):

$$\langle x', \mathbf{s}' | x, \mathbf{s} \rangle = \int d\mathbf{p} e^{i\mathbf{p} \cdot (x' - x)} \int dL e^{-mL} \int D\Lambda(s) e^{-i \int_0^L ds [p \cdot \Lambda \hat{t} - \sigma \text{tr}(\Lambda^{-1} \dot{\Lambda} M_{12})]}. \quad (128)$$

In practice, the path integral is found by combining the spin path integral eq. (98) and the spinless particle path integral eq. (115).

Let us now discuss in particular the spin- $\frac{1}{2}$ case in the spinor formulation, and show how the Dirac equation is recovered. We can do this promoting to the Lorentz group the connection between spinor and vector representations of the rotation group eq. (90). This is based on the transformation law of Dirac matrices, which connect the four-vector representation Λ of the Lorentz group with the corresponding spinor representation $T(\Lambda)$:

$$T(\Lambda^{-1}) \gamma^\mu T(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu. \quad (129)$$

Now, it is easy to show that given an unimodular vector v^μ , the spinor ψ such that

$$\psi^* \gamma^\mu \psi = v^\mu \quad (130)$$

satisfies the condition

$$v_\mu \gamma^\mu \psi = \psi \quad (131)$$

(in Euclidean space, in Minkowski space the spinor ψ^* must be replaced by $\bar{\psi} \equiv \psi^* \gamma^0$).

In our case, we associate to the one-particle state with normalized momentum e^μ the spinor $\psi[e^\mu]$ which satisfies the condition

$$p_\mu \gamma^\mu \psi = m\psi, \quad (132)$$

i.e. the Dirac equation. In practice, we can determine $\psi[e^\mu]$ by acting with the spinor representation $T(\Lambda)$ of the transformation Λ eq. (118)

$$\psi = T(\Lambda)\psi_0 \quad (133)$$

on the reference spinor ψ_0 such that $\psi_0^* \gamma^\mu \psi_0 = \hat{t}^\mu$, i.e. (using eq. (131) such that

$$\gamma^0 \psi_0 = \psi_0. \quad (134)$$

If one uses the so-called Dirac representation for the γ matrices, $\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ (where each entry is a 2×2 block), so

$$\psi = \begin{pmatrix} \phi \\ 0 \end{pmatrix}, \quad (135)$$

where ϕ is any two-component spinor.

The condition that the spin vector be given by s^μ fixes entirely the spinor (up to an overall $U(1)$ phase): if Λ is such that $\Lambda_\nu^\mu s_0^\nu = s^\mu$, then, choosing

according to eq. (119) $s_0^\nu = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, the spinor ψ is given by

$$\psi = T(\Lambda)\psi_0; \quad \phi_0 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (136)$$

It is easy to see that the spinor constructed in this way is an eigenstate of the projection of the Pauli-Lubanski operator along the spin vector $s^\mu = \frac{1}{2}n^\mu$:

$$W^\mu n_\mu \psi = \pm \frac{m}{2} \psi(p, s). \quad (137)$$

This is obvious in the rest frame, because then $W^\mu s_\mu = ms_i \epsilon^{ijk} \sigma_{jk} = m\mathbf{s} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ are Pauli matrices, and eq. (136) together with the relation between the spin vector and Pauli matrices eq. (90) implies that

$$\mathbf{s} \cdot \boldsymbol{\sigma} \phi = \pm \frac{1}{2} \phi. \quad (138)$$

In other words, in the rest frame $W^\mu s_\mu$ is just the standard spin operator and ϕ is the two-component spinor eq. (86) discussed in sec. 3.3. But since in the rest frame the low components of ψ eq. (135) vanish, this implies

$$W^\mu s_\mu \psi = \pm \frac{1}{2} m \psi. \quad (139)$$

In a generic frame, the four-vector s^μ is boosted by Λ , so

$$W^\mu n_\mu = W^\mu \Lambda^{-1}{}_\mu n_\mu = T(\Lambda) W^\mu T^{-1}(\Lambda) n_\mu, \quad (140)$$

but so is the spinor ψ in such a way that [eq. (133)] the eigenvector condition still holds:

$$T(\Lambda) W^\mu T^{-1}(\Lambda) n_\mu T(\Lambda) \psi_0 = \pm \frac{m}{2} T(\Lambda) \psi_0 \quad (141)$$

Let us now consider the propagator $K(p)$, i.e. momentum-space path integral, related by Fourier transformation to the path-integral eq. (128). We have found that in the spin- $\frac{1}{2}$ case, if the spinor representation is adopted, states along the path are instantaneous eigenstates of $e_\mu \gamma^\mu$, according to

eq. (132). It follows that momentum eigenstates, which are the boundary conditions to the momentum-space path-integral (i.e. states of definite e^μ) automatically satisfy the Dirac equation. Furthermore, the spinor states satisfy

$$\psi^* \gamma^\mu \psi = e^\mu, \quad (142)$$

i.e., e^μ is obtained by acting with γ^μ on the instantaneous spinor states along the path. But in sect. 3.3 we have proven [eq. (99)] that the expectation value of any function $F(\sigma)$ can be obtained by path-integration of the function $F(\Lambda)$ with a weight given by the spin action itself. Applying this in reverse, we see that averaging with the spin action produces the same result as taking matrix element of instantaneous (path-ordered) functions of γ^μ , where γ^μ is identified with e^μ thanks to eq. (142).

The propagator is therefore given by

$$\begin{aligned} K(p) &= \int dL e^{-mL} \int D\Lambda(s) e^{-i \int_0^L ds [p_\mu e^\mu - \sigma \text{tr}(\Lambda \dot{\Lambda} M_{12})]} \\ &= \int dL e^{-mL} e^{-i L p_\mu \gamma^\mu} \\ &= \frac{1}{p_\mu + m}, \end{aligned} \quad (143)$$

i.e. the usual Dirac form.

The link with Fermi statistics is understood by observing that the spin factor upon 2π rotation transforms as

$$\text{tr}(\Lambda^{-1} \dot{\Lambda} R M_{12} R^{-1}) = R_j^i \hat{z}^j \epsilon^{ijk} \text{tr}(\Lambda^{-1} \dot{\Lambda} M_{jk}) \quad (144)$$

so if $\sigma = \frac{1}{2}$ the path-integral eq. (128) acquires a phase $e^{i\pi} = -1$. In the more conventional approach, this follows from the anticommuting properties of the γ matrices, and it requires anticommuting (Grassmann) variables. In the geometric approach which we have followed this is not necessary, because the anticommutation properties follows automatically from the fact that physical states are localized on paths (so ordering along the path is enforced), and paths are given weights that transform nontrivially upon rotations. This provides an explicit realization of the general spin-statistics relation derived in section 2: once spin is obtained as a consequence of an interaction defined in configuration space, the link with statistics follows from the fact that particle interchange can be performed by 2π rotation.

Finally, it is interesting to observe that the dynamical coupling of spin and momentum which follows from the geometric interpretation of spin as a vector in the space which is orthogonal to momentum actually changes the nature of the sum over paths: the Hausdorff dimension of paths d_h that contribute to the regularized and renormalized Euclidean path integral in the continuum limit is not the same for Bose and Fermi particles [13]. The Hausdorff dimension relates the typical length scale L of paths which dominate the

propagator in the continuum limit to the momentum p which is propagated:

$$L \sim p^{d_H} \quad (145)$$

It can be proven that $d_H = 2$ for Bosons while $d_H = 1$ for Fermions [13]. A rough and ready way to see this is to compare the bosonic propagator eq. (117) and the fermionic propagator eq. (143): it appears that the scaling limit requires taking $Lm^a \sim \text{constant}$ with $a = 2$ for Bosons and $a = 1$ for Fermions. This means that Bosonic paths are coarser than Fermionic paths: Bosonic propagation is an ordinary random walk (like Brownian motion), whereas Fermionic propagation is a directed random walk, essentially because the spin interaction quenches fluctuations of the tangent vector to the path.

5 Conclusion

The discussion of spin presented in these lectures was rooted in quantum mechanics, and has used few field-theoretic concepts. Yet, we have been able to derive many results which usually require the full framework of relativistic quantum field theory: the spin-statistics connection, multivalued spin wave functions, the spin propagator, the Dirac equation. In fact, we have shown that the quantization of spin both in a nonrelativistic and a relativistic setting follows from general properties of the configuration space for orbits of the rotation group, viewed as a subgroup of the Galilei or Poincaré group, respectively. It thus appears that the standard field-theoretic approach is merely a convenient way of achieving the quantization of systems of elementary excitations which provide irreducible representations of the Galilei or Poincaré group, because field theory automatically combines quantum mechanics with the relevant symmetry group in a local, unitary way. Of course, the standard field-theoretic approach, with anticommuting variables and spinors, is by far more convenient for the sake of practical computations. However, we have attempted to show that the origin of the quantum field theoretic features of spin in the way symmetry is realized in quantum mechanics.

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